

Turbulence model of the cosmic structure

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Abstract

The Kolmogorov approach to turbulence is applied to the Burgers turbulence in the stochastic adhesion model of large-scale structure formation. As the perturbative approach to this model is unreliable, here is proposed a new, non-perturbative approach, based on a suitable formulation of Kolmogorov's scaling laws. This approach suggests that the power-law exponent of the matter density two-point correlation function is in the range 1–1.33, but it also suggests that the adhesion model neglects important aspects of the gravitational dynamics.

The large-scale structure of the universe is produced by the gravitational clustering of an initially homogeneous matter distribution. This process can be described by the Newtonian equations of motion of the matter fluid, written in comoving coordinates and in terms of the peculiar velocity and gravitational fields [1]. These equations are nonlinear and, although they can be linearized to describe the early growth of small perturbations of the initially homogeneous distribution, the actual structure formation takes place when the nonlinearity plays a major rôle, at the aptly called nonlinear stage of gravitational clustering. The nonlinearity of fluid mechanics also plays a major rôle in the phenomenon of turbulence and this is the cause of the difficulty in treating this phenomenon (often referred to as the “unsolved problem of classical physics”). Since turbulence is a key actor in many astrophysical scenarios, it is tempting to apply methods and ideas of turbulence to the study of large-scale structure formation.

An early attempt to apply Kolmogorov's scaling laws to the origin of galaxies was made by Weizsäcker [2], but his ideas have been since mostly restricted to intragalactic turbulence and have played no role in the study of the formation of clusters and superclusters of galaxies or the large-scale distribution of the dark matter. However, a popular model of large-scale structure formation, namely, the adhesion model [3, 4], is essentially a model of pressure-less turbulence, namely, the type of turbulence that occurs in strongly compressible flows and is usually called Burgers turbulence. As shown in this letter, the Kolmogorov approach to turbulence can be applied to cosmic Burgers turbulence, employing a suitable formulation of the adhesion model.

In cosmology, scaling laws for the velocity field, such as Kolmogorov's laws, are especially important if they can be related to scaling laws for the matter density field, because the positions of astronomical objects are more easily measurable than their velocities and indeed have been shown to follow scaling laws. The best known scaling laws in cosmology have been found in the distribution of galaxies [5, 6], but evidences of similar scaling laws in the dark-matter distribution are found in N -body cosmological simulations. To be precise, these simulations show that both the dark matter and the baryonic matter form a common *multifractal* “cosmic web” structure [7]. The cosmic web is precisely the type of structure predicted by the adhesion model [3, 4].

The cosmic web consists of sheets (Zeldovich “pancakes”), filaments and nodes, so it is indeed multifractal, in the sense that it is formed by objects of several dimensions, namely, two, one and zero dimensions. However, such a distribution needs not be scale invariant. Nevertheless, the actual cosmic web structure is, arguably, a self-similar multifractal [7]. The reason why the adhesion model does not predict a distribution with definite scaling properties is that the Burgers equation is *integrable*, in sharp contrast with the Navier-Stokes equation of incompressible turbulence. In other words, the *chaotic* properties of the latter are not present in the former, which nicely evolves the initial conditions. Therefore, the natural way of producing a self-similar cosmic web structure is to use self-similar initial conditions, namely, an initial Gaussian velocity distribution with a power-law power spectrum [8]. This type of distribution evolves to a self-similar cosmic web.

The self-similar cosmic-web solution is obtained from the exact integral of the Burgers equation in the zero-viscosity limit [8]. In this limit, the Burgers equation is indeed scale invariant, in the sense that simultaneous space and time scalings λx and $\lambda^{1-h}t$ and the induced scaling $\lambda^h \mathbf{u}$ leave the equation invariant. Hence, the solution fulfills the dynamical scaling law

$$\mathbf{u}(\mathbf{x}, t) = t^{h/(1-h)} \mathbf{u}(\mathbf{x}/t^{1/(1-h)}, 1), \quad (1)$$

that is, the solution at any t is obtained by scaling the solution at $t = 1$. This scaling is connected with a dynamical invariant, $u^{1/h}/x$, which can be identified with the specific dissipation rate ε for $h = 1/3$ (the Kolmogorov scaling). One can further deduce that there is a homogeneity scale $L(t) = t^{1/(1-h)}L(1)$, such that the cosmic web structure at time t has only formed on scales smaller than $L(t)$ whereas the initial homogeneous distribution stays on larger scales. The homogeneity scale L plays a similar rôle to that of the integral scale in Navier-Stokes turbulence.

The definition of matter density in the adhesion model is not unique, but there is an “analytically convenient” definition [9] that gives rise to a straightforward relation between velocity and density scaling laws, because the density field is expressed in terms of the velocity field:

$$\rho(\mathbf{x}) = \rho_0 \det [\delta_{ij} - \partial_i u_j(\mathbf{x})], \quad (2)$$

where ρ_0 is the constant initial density. This expression simplifies to

$$\delta\rho(\mathbf{x}) = \rho(\mathbf{x}) - \rho_0 = -\rho_0 \partial_i u^i(\mathbf{x}) \quad (3)$$

in the linear regime, $|\delta\rho| \ll \rho_0$. In the nonlinear regime, as shock waves form, $\mathbf{u}(\mathbf{x})$ becomes discontinuous and $\partial_i u_j \rightarrow -\infty$. The shock waves are also *caustics* [3, 4], where matter accumulates and $\rho \rightarrow \infty$, according to Eq. (2).

Unfortunately, the above-explained approach has obvious shortcomings: (i) the initial power spectrum is not a power law; (ii) the adhesion model is just a simplified model of the gravitational dynamics, which actually *is* chaotic; and (iii) the adhesion of matter into caustics is considered as an inelastic collision but the dissipated energy is not accounted for. In fact, the dissipation in the gravitational dynamics is linked to its chaotic nature: self-gravitating systems tend to virial equilibrium, which is independent of the initial conditions and implies that entropy grows in the process (this process is usually called virialization). Therefore, it is reasonable to supplement the adhesion model with a “noise”, which reverts the lost kinetic energy, on the one hand, and makes the long-time evolution of the velocities independent of the initial conditions, on the other hand. This *stochastic adhesion model* possesses an attractor characterized by a dynamical scaling that is independent of the initial conditions, unlike the one defined by Eq. (1). The resulting stationary state, in which the energy injected on scales $> L$ is dissipated at constant rate ε at the Kolmogorov scale, is analogous to the stationary state of incompressible turbulence. However, in Burgers turbulence, the dissipation takes place in caustics and has more spatial variation than in incompressible turbulence, producing strong *intermittency*. Intermittency leads to deviations from Kolmogorov’s scaling for higher order correlation functions. A beautiful exposition of Kolmogorov’s ideas and of intermittency is given by Frisch [10].

The stochastic Burgers equation is well studied, since it appears, in terms of the velocity potential, as an equation for surface growth (the surface’s height is given by the potential). The equation for the potential, called the KPZ equation, includes a Gaussian noise with power spectrum $D(k, \omega)$. It has been studied with renormalized perturbation theory [11]. With this method, the types of noise that give rise to dynamical scaling are determined as fixed points of the dynamical renormalization group. For white noise, in three spatial dimensions, the nontrivial fixed point is repulsive, so the nonlinear term of the Burgers equation is *irrelevant* (in the renormalization group sense) and the viscous term dominates in the perturbative stationary state. Therefore, turbulence can only occur in the strong-coupling, non-perturbative regime. The addition of “colored” noise with power-law spectrum $D(k) = Dk^{-2\rho}$ [11] does not improve the situation: there can be several fixed points, but, in three dimensions, only the trivial fixed point is stable and only if ρ is small; otherwise, it becomes a saddle point. This means that a noise with sufficient power on large scales inevitably leads to a strong-coupling stationary state.

The method of Medina et al [11] has been adapted to the cosmology setting by Domínguez et al [12]. They consider noise with power-law spatial and *temporal* correlations, $D(k, \omega) = Dk^{-2\rho}\omega^{-2\theta}$, and one more coupling (a sort of QFT “mass”). The corresponding renormalization group equations have several fixed points, but only one is stable. A choice of ρ and θ in certain ranges yields exponents for the power-law velocity-potential correlation function such that the corresponding exponents of the density correlation function, obtained through Eq. (3), fit the range of measured exponents γ of the two-point correlation function

of galaxies. This intriguing derivation of γ has several questionable aspects, besides the ad-hoc choice of ρ and θ . First, Galilean invariance is broken, as the existence of the stable fixed point demands a non-zero θ [12]. Second, Eq. (3) is only valid in the linear regime, in principle. Third, the results, apart from the values of γ , are also questionable: Regarding the values of the couplings at the fixed-point, the strength D of the correlated part of the noise (dominant for small k) turns out to be negative. Furthermore, the “mass” scale is non-vanishing, so the stable fixed point does not seem to correspond to a scale invariant stationary state.

At any rate, one can argue, on general grounds, that perturbation theory (especially, the one-loop approximation) is not the right approach to Burgers turbulence. The effective coupling constants in the renormalization group equations have the generic expression $\lambda^2 D/\nu^3$ (except the “mass”), where λ is the nonlinear coupling constant (to be set to the value of unity), D is a noise strength, and ν the viscosity. Therefore, as the nonlinear term dominates in the inertial range, the coupling must be strong. More precisely, the given expression implies that the coupling constants are actually proportional to the cube of the Reynolds number, which has to be a very large number, making perturbation theory unreliable.

Therefore, one must resort to non-perturbative methods. Standard non-perturbative methods in turbulence are the closure approaches, in which the hydrodynamical hierarchy of equations for statistical moments is closed at some order by assuming a relation between the moments of the corresponding order and lower order ones. There is a similar closure approach in cosmology, based on the BBGKY hierarchy [1]. This a second-order closure and it is consistent with a scaling ansatz for the two-point correlation functions, with just one power-law exponent, but this number remains undetermined, unless a connection with an initial power-law power spectrum of perturbations is assumed. As we avoid this connection, we prefer to follow the traditional non-perturbative methods in turbulence. They are based on reasonable assumptions, the simplest ones being Kolmogorov’s universality assumptions, namely, homogeneity, isotropy, and scaling laws for the moments of longitudinal velocity increments [10]. These laws state that

$$\langle (\delta \mathbf{u} \cdot \mathbf{r}/r)^n \rangle \propto (\varepsilon r)^{n/3}, \quad (4)$$

where $\delta \mathbf{u} = \mathbf{u}(\mathbf{x} + \mathbf{r}/2) - \mathbf{u}(\mathbf{x} - \mathbf{r}/2)$ and $n \in \mathbb{N}$. A general form of these scaling laws, suitable for introducing the effect of intermittency, is

$$\langle |\delta \mathbf{u}|^q \rangle = A r^{\zeta(q)}, \quad (5)$$

where $q \in \mathbb{R}$, and A does not depend on r . The effect of intermittency is given by the function $\zeta(q)$ as explained in the following.

Kolmogorov’s scaling laws are justified by employing the hierarchy of hydrodynamical equations, in particular, the second-order one, called the Karman-Howarth-Monin equation [10]. A version of this equation is valid for Burgers turbulence. An illuminating derivation of the equation has been given by Polyakov [13] in the one-dimensional case. Polyakov realizes that, in the equation for $\partial_t u^2$, the dissipation in the inertial range arises as a

field-theory anomaly, due to the non-differentiability of the velocity field. The form of the anomaly can be found by employing a point-splitting method. In the three-dimensional case, the calculation is more involved but it yields the simple result:

$$\partial_t u^2(\mathbf{x}) = -u^i(\mathbf{x}) \frac{\partial u^2(\mathbf{x})}{\partial x^i} + \frac{1}{2} \lim_{r \rightarrow 0} \frac{\partial}{\partial r^i} (\delta u^i \delta \mathbf{u}^2). \quad (6)$$

The last term is the anomaly $a(\mathbf{x})$, which would vanish if $\mathbf{u}(\mathbf{x})$ were differentiable. Remarkably, to derive Eq. (6), we do not need homogeneity or isotropy. Anyway, homogeneity and isotropy are part of Kolmogorov's universality assumptions and are natural in cosmology. From Eq. (6), one deduces that the average dissipation in the steady state is $\varepsilon = -\langle a \rangle/2$. This closure relation is an exact formulation of the $n = 3$ case of the scaling laws (4) for Burgers turbulence, analogous to Kolmogorov's "4/5" law of incompressible turbulence [10]. Therefore, in Eq. (5), $\zeta(3) = 1$ (assuming that ε is well defined in the limit $\nu \rightarrow 0$).

If the probability $P(\delta \mathbf{u})$ were Gaussian, then $\zeta(q) \propto q$, and necessarily $\zeta(q) = q/3$, as in Eq. (4). However, intermittency manifests itself in a slower growth of $\zeta(q)$ for $q > 3$ [10]. In one dimension and with power-law correlated noise, the extent of intermittency in Burgers turbulence depends on the noise exponent, but it is always so strong that the maximum of $\zeta(q)$ is $\zeta = 1$ [14, especially, Fig. 2]. Generalizing the results of Hayot and Jayaprakash [14] to three dimensions, in terms of the KPZ noise exponent ρ , the value $\rho = 5/2$ is such that the noise strength D has the dimensions of ε and the (Burgers) noise correlation function is proportional to $\log r$. This leads to the Kolmogorov scaling law $\zeta(q) = q/3$ for $q \leq 3$ [14]. The limit of the noise correlation function as $r \rightarrow 0$, equal to ε , diverges for $\rho < 5/2$ and is ill-defined for $\rho > 5/2$, in the limits $\nu \rightarrow 0$ or $L \rightarrow \infty$, respectively. In other words, $\varepsilon = \int_0^\infty k^2 D(k) d^3k$ diverges at $k = \infty$ or at $k = 0$ and therefore is not universal. However, the r -dependent part of the noise correlation function is universal and proportional to $r^{2\rho-5}$ if $3/2 < \rho < 7/2$, $\rho \neq 5/2$. The values $5/2 < \rho < 7/2$ correspond to large-scale forcing, such that $\zeta(q) = 1$ if $q \geq 3$, but $2/3 < \zeta(2) < 1$. For $\rho > 7/2$, the r -dependent part of the noise correlation function is not universal and depends on scales $> L$; that is to say, it depends on the initial conditions. Furthermore, $\zeta(q) = 1$ for $q \geq 2$, in this case. In this range of ρ , the stochastic adhesion model is presumably equivalent to the ordinary adhesion model with self-similar initial conditions. Therefore, the interesting range for the cosmic structure is $\rho \in (5/2, 7/2)$. Note that the noise correlation function does not need to be a power law: higher powers of r are not relevant in the inertial range and, besides, are not universal.

Our next step is to calculate the density correlation function from Eq. (2), assuming Eq. (5) with $\zeta(q) = 1$ for $q \geq 3$ and $2/3 < \zeta(2) < 1$. The expansion of the determinant in Eq. (2) yields:

$$\delta\rho/\rho_0 = -\partial_i u^i + (\partial_1 u^1 \partial_2 u^2 - \partial_1 u^2 \partial_2 u^1 + \partial_1 u^1 \partial_3 u^3 - \partial_1 u^3 \partial_3 u^1 + \partial_2 u^2 \partial_3 u^3 - \partial_2 u^3 \partial_3 u^2) + O(\partial u)^3.$$

While Eq. (3), of $O(\partial u)$, is valid in the linear regime, we have to consider the formation of caustics. Caustics are due to the blowing up of the eigenvalues of the matrix $\partial_i u^j = \partial_{ij} \phi$,

where ϕ is the velocity potential (\mathbf{u} is discontinuous). If only one eigenvalue diverges, the collapse is one-dimensional and a sheet forms, so Eq. (3) is justified. For filaments or nodes, more terms are necessary.

The reduced two-point correlation function of the density is

$$\langle \delta\rho(\mathbf{r}) \delta\rho(\mathbf{0}) \rangle / \rho_0^2 = \langle \partial_i u^i(\mathbf{r}) \partial_j u^j(\mathbf{0}) \rangle - 2c(r) + \langle O(\partial u)^4 \rangle + \dots, \quad (7)$$

where

$$c(r) = \langle (\partial_1 u^1 \partial_2 u^2 - \partial_1 u^2 \partial_2 u^1 + \partial_1 u^1 \partial_3 u^3 - \partial_1 u^3 \partial_3 u^1 + \partial_2 u^2 \partial_3 u^3 - \partial_2 u^3 \partial_3 u^2)(\mathbf{r}) \partial_j u^j(\mathbf{0}) \rangle.$$

We have shown explicitly only terms up to $O(\partial u)^3$, because the other terms do not require any calculation, as we now explain. The functions on the right-hand side of Eq. (7) are power-laws of r , each one with a characteristic exponent $-\gamma$ that can be deduced from Eq. (5); namely, $-\gamma = \zeta(n) - n$ for $\langle O(\partial u)^n \rangle$. Given that $\zeta(n) = 1$ for $n = 3, 4, 5, 6$, we have $\gamma = 2, 3, 4, 5$, respectively. However, the maximal value is $\gamma = 3$, which is the value for a Poisson distribution (shot-noise) term: this term can appear as either $\delta(\mathbf{r})$ or r^{-3} (see, e.g., [15]). As $\gamma = 3$ is reached for $n \geq 4$, we only need to consider the cases $n = 3$ and $n = 2$.

To explicitly calculate $c(r) \propto r^{-2}$, it is useful to express it as $c(r) = \Delta g(r)$, where

$$g(r) = \langle (\partial_{11}\phi \partial_{22}\phi - \partial_{12}\phi \partial_{21}\phi + \partial_{11}\phi \partial_{33}\phi - \partial_{13}\phi \partial_{31}\phi + \partial_{22}\phi \partial_{33}\phi - \partial_{23}\phi \partial_{32}\phi)(\mathbf{r}) \phi(\mathbf{0}) \rangle.$$

This function is a dimensionless scalar, so it must be a constant. Therefore, $c(r) = 0$ and the $\gamma = 2$ contribution vanishes.

In the end, the relevant contribution to the density two-point correlation function is due to the velocity two-point correlation function. The corresponding exponent is $\gamma = 2 - \zeta(2)$. Since $2/3 < \zeta(2) < 1$, we obtain $1 < \gamma < 4/3$. The Kolmogorov scaling $\zeta(2) = 2/3$ yields the upper bound, $\gamma = 4/3 \simeq 1.33$. The range of values of γ obtained from galaxy surveys or N -body cosmological simulations is (mostly) in the interval $(1, 2)$ [5, 6, 7]. However, the classic value $\gamma = 1.7$, which still stands [6, 7], is larger than $4/3$. Nevertheless, a specific methodology for the analysis of galaxy catalogs [5] yields values of γ in the interval $(1, 1.3)$.

To obtain values of γ in the interval $(4/3, 2)$, we could take $\rho \in (3/2, 5/2)$. Then, the Kolmogorov scale could not be set to zero, so it should be kept and, preferably, identified with a physical scale. In the gravitational dynamics, there is no intrinsic small scale, but there are small scales in the initial conditions. In N -body cosmological simulations, the most suitable small scale is the scale of gravitational smoothing. In any case, if we were to take $\rho \in (3/2, 5/2)$, then $\zeta(3) < 1$, so $c(r)$ would not vanish. When the density correlation function is the sum of different powers of r , the most singular term dominates in the nonlinear domain $r \ll L$ (the inertial range). The most singular term is, of course, the Poisson term, but it must be discarded [15]. The next singular component is $c(r)$, although it vanishes for $\rho > 5/2$. In contrast, for $\rho < 5/2$, $c(r)$ would not vanish and would lead to a $\gamma > 2$, instead of a $\gamma \in (4/3, 2)$.

With $1 \leq \gamma < 4/3$, the density correlation function is dominated by sheets. Interestingly, the analysis of N -body cosmological simulations leads to a similar conclusion: the

bulk of mass belongs to sheets [7]. However, to fully understand the rôle of the three types of cosmic-web singularities, namely, sheets, filaments and nodes, one must go beyond the scope of the adhesion model, because the three types of singularities are very different in regard to the gravitational dynamics. The accumulation of matter in sheets leads to density singularities, but the gravitational potential stays finite. In contrast, filaments and nodes are gravitational singularities as well as density singularities, so their formation involves the dissipation of an infinite amount of energy. Therefore, it is not surprising that the analysis of N -body cosmological simulations [7] shows that the spectrum of local dimensions α is cut off at $\alpha = 1$, which is precisely the local dimension of filaments. For filaments, the gravitational potential has just a logarithmic singularity, which is milder than the r^{-1} singularity of nodes and, hence, involves less dissipation.

At any rate, gravitational singularities cannot be described in a Newtonian framework and need the Theory of General Relativity. In this theory, the energy dissipated in, for example, the formation of a point singularity as a black hole is finite, namely, it is given by the well-studied black-hole entropy. Plausibly, a good part of the gravitational energy dissipation at the (cosmic) Kolmogorov scale can be attributed to the formation and growth of super-massive black holes, which occur due to dissipative processes in the dark matter and, preferentially, in the baryonic matter. However, the formation and growth of black holes or other relativistic gravitational singularities is beyond the scope of the adhesion model and even beyond the scope of (state-of-the-art) N -body cosmological simulations.

In conclusion, a hydrodynamic closure approach to three-dimensional Burgers turbulence leads to Kolmogorov's scaling laws, although in a general form compatible with the presence of intermittency. These scaling laws can be applied to the stochastic adhesion model of the cosmic structure, in particular, to the determination of the density two-point correlation function. The result is in partial agreement with the two-point correlation function obtained from the distribution of galaxies and from N -body simulations but suggests that the adhesion model underestimates the contribution of low dimensional singularities (filaments and nodes) to energy dissipation, whereas N -body simulations overestimate it. It is probably necessary to have a better modeling of small-scale dissipative processes, and this modeling may require ingredients from general relativity.

References

- [1] P.E.J. Peebles, *The large-scale structure of the universe* (Princeton University, Princeton, NJ, 1980)
- [2] C. F. von Weizsäcker, *The Astrophysical Journal* **114**, 165 (1951)
- [3] S.N. Gurbatov and A.I. Saichev, *Radiophysics and Quantum Electronics* **27**, 303–313 (1984)
- [4] S.F. Shandarin and Ya.B. Zeldovich, *Rev. Mod. Phys.* **61**, 185–220 (1989)
- [5] F. Sylos Labini, M. Montuori, and L. Pietronero, *Phys. Rep.* **293**, 61 (1998)

- [6] B.J. Jones, V.J. Martínez, E. Saar, and V. Trimble, *Rev. Mod. Phys.* **76**, 1211 (2004)
- [7] J. Gaite, *The Astrophysical Journal* **658**, 11–24 (2007); *JCAP* **03**, 006 (2010)
- [8] M. Vergassola, B. Dubrulle, U. Frisch, and A. Noullez, *Astron. Astrophys.* **289**, 325 (1994)
- [9] A.I. Saichev and W.A. Woyczynski, *SIAM J. Appl. Math.* **56**, 1008–1038 (1996)
- [10] Frisch U., *Turbulence: The Legacy of A.N. Kolmogorov* (Cambridge University Press, 1995)
- [11] E. Medina, T. Hwa, M. Kardar and Y.-Ch. Zhang, *Phys. Rev. A* **39**, 3053–3075 (1989)
- [12] A. Domínguez, D. Hochberg, J.M. Martín-García, J. Pérez-Mercader, and L.S. Schulman, *Astron. Astrophys.* **344**, 27 (1999)
- [13] A.M. Polyakov, *Phys. Rev. E* **52**, 6183–6188 (1995)
- [14] F. Hayot, and C. Jayaprakash, *International Journal of Modern Physics B* **14**, 1781–1800 (2000)
- [15] J. Gaite, A. Domínguez, and J. Pérez-Mercader, *Astrophys. J. Lett.* **522**, L5 (1999)